4.2

Mean Value Theorem

Mean Value Theorem

The Mean Value Theorem connects the average rate of change of a function over an interval with the instantaneous rate of change of the function at a point within the interval. Its powerful corollaries lie at the heart of some of the most important applications of the calculus.

The theorem says that somewhere between points $A$ and $B$ on a differentiable curve, there is at least one tangent line parallel to chord $AB$ (Figure 4.10).

THEOREM 3  Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$, then there is at least one point $c$ in $(a, b)$ at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$  

The hypotheses of Theorem 3 cannot be relaxed. If they fail at even one point, the graph may fail to have a tangent parallel to the chord. For instance, the function $f(x) = |x|$ is continuous on $[-1, 1]$ and differentiable at every point of the interior $(-1, 1)$ except $x = 0$. The graph has no tangent parallel to chord $AB$ (Figure 4.11a). The function $g(x) = \text{int } x$ is differentiable at every point of $(1, 2)$ and continuous at every point of $[1, 2]$ except $x = 2$. Again, the graph has no tangent parallel to chord $AB$ (Figure 4.11b).

The Mean Value Theorem is an existence theorem. It tells us the number $c$ exists without telling how to find it. We can sometimes satisfy our curiosity about the value of $c$ but the real importance of the theorem lies in the surprising conclusions we can draw from it.

EXAMPLE 1  Exploring the Mean Value Theorem

Show that the function $f(x) = x^2$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find a solution $c$ to the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

on this interval.

continued
SOLUTION
The function \( f(x) = x^2 \) is continuous on \([0, 2]\) and differentiable on \((0, 2)\). Since \( f(0) = 0 \) and \( f(2) = 4 \), the Mean Value Theorem guarantees a point \( c \) in the interval \((0, 2)\) for which

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

\[
2c = \frac{f(2) - f(0)}{2 - 0} = 2 \quad f'(x) = 2x
\]

\[
c = 1.
\]

Interpret The tangent line to \( f(x) = x^2 \) at \( x = 1 \) has slope 2 and is parallel to the chord joining \( A(0, 0) \) and \( B(2, 4) \) (Figure 4.12).

Example 2 Exploring the Mean Value Theorem

Explain why each of the following functions fails to satisfy the conditions of the Mean Value Theorem on the interval \([-1, 1]\).

(a) \( f(x) = \sqrt{x^2} + 1 \)

(b) \( f(x) = \begin{cases} x^3 + 3 & \text{for } x < 1 \\ x^2 + 1 & \text{for } x \geq 1 \end{cases} \)

SOLUTION
(a) Note that \( \sqrt{x^2} + 1 = |x| + 1 \), so this is just a vertical shift of the absolute value function, which has a nondifferentiable “corner” at \( x = 0 \). (See Section 3.2.) The function \( f \) is not differentiable on \((-1, 1)\).

(b) Since \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} x^3 = 4 \) and \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} x^2 + 1 = 2 \), the function has a discontinuity at \( x = 1 \). The function \( f \) is not continuous on \([-1, 1]\).

If the two functions given had satisfied the necessary conditions, the conclusion of the Mean Value Theorem would have guaranteed the existence of a number \( c \) in \((-1, 1)\) such that \( f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = 0 \). Such a number \( c \) does not exist for the function in part (a), but one happens to exist for the function in part (b) (Figure 4.13).

Figure 4.13 For both functions in Example 2, \( \frac{f(1) - f(-1)}{1 - (-1)} = 0 \) but neither function satisfies the conditions of the Mean Value Theorem on the interval \([-1, 1]\). For the function in Example 2(a), there is no number \( c \) such that \( f'(c) = 0 \). It happens that \( f'(0) = 0 \) in Example 2(b).
EXAMPLE 3 Applying the Mean Value Theorem
Let \( f(x) = \sqrt{1 - x^2} \), \( A = (-1, f(-1)) \), and \( B = (1, f(1)) \). Find a tangent to \( f \) in the interval \((-1, 1)\) that is parallel to the secant \( AB \).

SOLUTION
The function \( f \) (Figure 4.14) is continuous on the interval \([-1, 1]\) and \( f \) is defined on the interval \((-1, 1)\). The function is not differentiable at \( x = -1 \) and \( x = 1 \), but it does not need to be for the theorem to apply. Since \( f(-1) = f(1) = 0 \), the tangent we are looking for is horizontal. We find that \( f' = 0 \) at \( x = 0 \), where the graph has the horizontal tangent \( y = 1 \).

Now try Exercise 9.

Physical Interpretation
If we think of the difference quotient \( (f(b) - f(a))/(b - a) \) as the average change in \( f \) over \([a, b]\) and \( f'(c) \) as an instantaneous change, then the Mean Value Theorem says that the instantaneous change at some interior point must equal the average change over the entire interval.

EXAMPLE 4 Interpreting the Mean Value Theorem
If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is \( 352/8 = 44 \) ft/sec, or 30 mph. At some point during the acceleration, the theorem says, the speedometer must read exactly 30 mph (Figure 4.15).

Now try Exercise 11.

Increasing and Decreasing Functions
Our first use of the Mean Value Theorem will be its application to increasing and decreasing functions.

DEFINITIONS Increasing Function, Decreasing Function
Let \( f \) be a function defined on an interval \( I \) and let \( x_1 \) and \( x_2 \) be any two points in \( I \).

1. \( f \) increases on \( I \) if \( x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \).
2. \( f \) decreases on \( I \) if \( x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \).

The Mean Value Theorem allows us to identify exactly where graphs rise and fall. Functions with positive derivatives are increasing functions; functions with negative derivatives are decreasing functions.

COROLLARY 1 Increasing and Decreasing Functions
Let \( f \) be continuous on \([a, b]\) and differentiable on \((a, b)\).

1. If \( f'' > 0 \) at each point of \((a, b)\), then \( f \) increases on \([a, b]\).
2. If \( f'' < 0 \) at each point of \((a, b)\), then \( f \) decreases on \([a, b]\).
Mean Value Theorem

Determining Where Graphs Rise or Fall

The graph of \( f \) is increasing on \( (a, b) \) (if \( f' > 0 \)), or \( f \) is decreasing on \( (a, b) \) (if \( f' < 0 \)).

**Example 5** Determining Where Graphs Rise or Fall

The function \( f(x) = x^2 \) (Figure 4.16) is

(a) decreasing on \( (-\infty, 0] \) because \( y' = 2x < 0 \) on \( (-\infty, 0) \).

(b) increasing on \( [0, \infty) \) because \( y' = 2x > 0 \) on \( (0, \infty) \).

Now try Exercise 15.

**Example 6** Determining Where Graphs Rise or Fall

Where is the function \( f(x) = x^3 - 4x \) increasing and where is it decreasing?

**Solution**

**Solve Graphically** The graph of \( f \) in Figure 4.17 suggests that \( f \) is increasing from \( -\infty \) to the \( x \)-coordinate of the local maximum, decreasing between the two local extrema, and increasing again from the \( x \)-coordinate of the local minimum to \( \infty \). This information is supported by the superimposed graph of \( f'(x) = 3x^2 - 4 \).

**Confirm Analytically** The function is increasing where \( f'(x) > 0 \).

\[
3x^2 - 4 > 0
\]

\[
x^2 > \frac{4}{3}
\]

\[
x < -\sqrt{\frac{4}{3}} \quad \text{or} \quad x > \sqrt{\frac{4}{3}}
\]

The function is decreasing where \( f'(x) < 0 \).

\[
3x^2 - 4 < 0
\]

\[
x^2 < \frac{4}{3}
\]

\[-\sqrt{\frac{4}{3}} < x < \sqrt{\frac{4}{3}}
\]

In interval notation, \( f \) is increasing on \( (-\infty, -\sqrt{4/3}) \), decreasing on \( [-\sqrt{4/3}, \sqrt{4/3}] \), and increasing on \( [\sqrt{4/3}, \infty) \).

Now try Exercise 27.

**Other Consequences**

We know that constant functions have the zero function as their derivative. We can now use the Mean Value Theorem to show conversely that the only functions with the zero function as derivative are constant functions.

**Corollary 2** Functions with \( f' = 0 \) are Constant

If \( f'(x) = 0 \) at each point of an interval \( I \), then there is a constant \( C \) for which \( f(x) = C \) for all \( x \) in \( I \).
Proof Our plan is to show that \( f(x_1) = f(x_2) \) for any two points \( x_1 \) and \( x_2 \) in \( I \). We can assume the points are numbered so that \( x_1 < x_2 \). Since \( f \) is differentiable at every point of \([x_1, x_2]\), it is continuous at every point as well. Thus, \( f \) satisfies the hypotheses of the Mean Value Theorem on \([x_1, x_2]\). Therefore, there is a point \( c \) between \( x_1 \) and \( x_2 \) for which

\[
f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.
\]

Because \( f'(c) = 0 \), it follows that \( f(x_1) = f(x_2) \). ■

We can use Corollary 2 to show that if two functions have the same derivative, they differ by a constant.

**COROLLARY 3** Functions with the Same Derivative Differ by a Constant

If \( f'(x) = g'(x) \) at each point of an interval \( I \), then there is a constant \( C \) such that \( f(x) = g(x) + C \) for all \( x \) in \( I \).

**Proof** Let \( h = f - g \). Then for each point \( x \) in \( I \),

\[
h'(x) = f'(x) - g'(x) = 0.
\]

It follows from Corollary 2 that there is a constant \( C \) such that \( h(x) = C \) for all \( x \) in \( I \). Thus, \( h(x) = f(x) - g(x) = C \), or \( f(x) = g(x) + C \). ■

We know that the derivative of \( f(x) = x^2 \) is \( 2x \) on the interval \((-\infty, \infty)\). So, any other function \( g(x) \) with derivative \( 2x \) on \((-\infty, \infty)\) must have the formula \( g(x) = x^2 + C \) for some constant \( C \).

**EXAMPLE 7** Applying Corollary 3

Find the function \( f(x) \) whose derivative is \( \sin x \) and whose graph passes through the point \((0, 2)\).

**SOLUTION**

Since \( f \) has the same derivative as \( g(x) = -\cos x \), we know that \( f(x) = -\cos x + C \), for some constant \( C \). To identify \( C \), we use the condition that the graph must pass through \((0, 2)\). This is equivalent to saying that

\[
\begin{align*}
f(0) &= 2 \\
-\cos (0) + C &= 2 \\
-1 + C &= 2 \\
C &= 3.
\end{align*}
\]

The formula for \( f \) is \( f(x) = -\cos x + 3 \). Now try Exercise 35.

In Example 7 we were given a derivative and asked to find a function with that derivative. This type of function is so important that it has a name.

**DEFINITION** Antiderivative

A function \( F(x) \) is an antiderivative of a function \( f(x) \) if \( F'(x) = f(x) \) for all \( x \) in the domain of \( f \). The process of finding an antiderivative is antidifferentiation.
We know that if \( f \) has one antiderivative \( F \) then it has infinitely many antiderivatives, each differing from \( F \) by a constant. Corollary 3 says these are all there are. In Example 7, we found the particular antiderivative of \( \sin x \) whose graph passed through the point \((0, 2)\).

**EXAMPLE 8  Finding Velocity and Position**

Find the velocity and position functions of a body falling freely from a height of 0 meters under each of the following sets of conditions:

(a) The acceleration is 9.8 \( m/sec^2 \) and the body falls from rest.
(b) The acceleration is 9.8 \( m/sec^2 \) and the body is propelled downward with an initial velocity of 1 \( m/sec \).

**SOLUTION**

(a) **Falling from rest.** We measure distance fallen in meters and time in seconds, and assume that the body is released from rest at time \( t = 0 \).

**Velocity:** We know that the velocity \( v(t) \) is an antiderivative of the constant function 9.8. We also know that \( g(t) = 9.8t \) is an antiderivative of 9.8. By Corollary 3,

\[
v(t) = 9.8t + C
\]

for some constant \( C \). Since the body falls from rest, \( v(0) = 0 \). Thus,

\[
9.8(0) + C = 0 \quad \text{and} \quad C = 0.
\]

The body’s velocity function is \( v(t) = 9.8t \).

**Position:** We know that the position \( s(t) \) is an antiderivative of 9.8. We also know that \( h(t) = 4.9t^2 \) is an antiderivative of 9.8. By Corollary 3,

\[
s(t) = 4.9t^2 + C
\]

for some constant \( C \). Since \( s(0) = 0 \),

\[
4.9(0)^2 + C = 0 \quad \text{and} \quad C = 0.
\]

The body’s position function is \( s(t) = 4.9t^2 \).

(b) **Propelled downward.** We measure distance fallen in meters and time in seconds, and assume that the body is propelled downward with velocity of 1 \( m/sec \) at time \( t = 0 \).

**Velocity:** The velocity function still has the form \( 9.8t + C \), but instead of being zero, the initial velocity (velocity at \( t = 0 \)) is now 1 \( m/sec \). Thus,

\[
9.8(0) + C = 1 \quad \text{and} \quad C = 1.
\]

The body’s velocity function is \( v(t) = 9.8t + 1 \).

**Position:** We know that the position \( s(t) \) is an antiderivative of \( 9.8t + 1 \). We also know that \( k(t) = 4.9t^2 + t \) is an antiderivative of \( 9.8t + 1 \). By Corollary 3,

\[
s(t) = 4.9t^2 + C
\]

for some constant \( C \). Since \( s(0) = 0 \),

\[
4.9(0)^2 + 0 + C = 0 \quad \text{and} \quad C = 0.
\]

The body’s position function is \( s(t) = 4.9t^2 + t \).  

*Now try Exercise 43.*
Quick Review 4.2  (For help, go to Sections 1.2, 2.3, and 3.2.)

In Exercises 1 and 2, find exact solutions to the inequality.
1. \(2x^2 - 6 < 0\) \(-\sqrt{3}, \sqrt{3}\) \(3x^2 - 6 > 0\) \(-\infty, -\sqrt{2}\) \(\cup\) \((\sqrt{2}, \infty)\)

In Exercises 3–5, let \(f(x) = \sqrt{8 - 2x^2}\).
3. Find the domain of \(f\). \([-2, 2]\)
4. Where is \(f\) continuous? For all \(x\) in its domain, or, \([-2, 2]\)
5. Where is \(f\) differentiable? On \((-2, 2)\)

In Exercises 6–8, let \(f(x) = \frac{x}{x^2 - 1}\).
6. Find the domain of \(f\). \(x \neq \pm 1\)
7. Where is \(f\) continuous? For all \(x\) in its domain, or, for all \(x \neq \pm 1\)
8. Where is \(f\) differentiable? For all \(x\) in its domain, or, for all \(x \neq \pm 1\)

In Exercises 9 and 10, find \(C\) so that the graph of the function \(f\) passes through the specified point.
9. \(f(x) = -2x + C\), \((-2, 7)\) \(C = 3\)
10. \(g(x) = x^2 + 2x + C\), \((1, -1)\) \(C = -4\)

Section 4.2 Exercises

In Exercises 1–8, (a) state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and (b) if it does, find each value of \(c\) in the interval \((a, b)\) that satisfies the equation \(f'(c) = \frac{f(b) - f(a)}{b - a}\).

1. \(f(x) = x^2 + 2x - 1\) on \([0, 1]\)
2. \(f(x) = x^{2/3}\) on \([0, 1]\)
3. \(f(x) = x^{1/3}\) on \([-1, 1]\) No. There is a vertical tangent at \(x = 0\).
4. \(f(x) = |x - 1|\) on \([0, 4]\) No. There is a corner at \(x = 1\).
5. \(f(x) = \sin^{-1}x\) on \([-1, 1]\)
6. \(f(x) = \ln(x - 1)\) on \([2, 4]\)
7. \(f(x) = \begin{cases} \cos x, & 0 \leq x < \pi/2 \\ \sin x, & \pi/2 \leq x \leq \pi \end{cases}\) on \([0, \pi]\)
   No. The split function is discontinuous at \(x = \pi/2\).
8. \(f(x) = \begin{cases} \sin^{-1}x, & -1 \leq x < 1 \\ x/2 + 1, & 1 \leq x \leq 3 \end{cases}\) on \([-1, 3]\)
   No. The split function is discontinuous at \(x = 1\).

In Exercises 9 and 10, the interval \(a \leq x \leq b\) is given. Let \(A = (a, f(a))\) and \(B = (b, f(b))\). Write an equation for
(a) the secant line \(AB\).
(b) a tangent line to \(f\) in the interval \((a, b)\) that is parallel to \(AB\).
9. \(f(x) = x + 1/x\), \(0.5 \leq x \leq 2\) \((a) y = \frac{5}{2}\) \((b) y = 2\)
10. \(f(x) = \sqrt{x - 1}\), \(1 \leq x \leq 4\) See page 204.

11. Speeding  A trucker handed in a ticket at a toll booth showing that in 2 h she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?
12. Temperature Change  It took 20 sec for the temperature to rise from 0°F to 10°F when a thermometer was taken from a freezer and placed in boiling water. Explain why at some moment in that interval the mercury was rising at exactly 10.6°F/sec.
13. Triremes  Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 h. Explain why at some point during this feat the trireme’s speed exceeded 7.5 knots (sea miles per hour).
14. Running a Marathon  A marathoner ran the 26.2-mi New York City Marathon in 2.2 h. Show that at least twice, the marathoner was running at exactly 11 mph.

In Exercises 15–22, use analytic methods to find (a) the local extrema, (b) the intervals on which the function is increasing, and (c) the intervals on which the function is decreasing.
15. \(f(x) = 5x - x^2\) See page 204.
16. \(g(x) = x^2 - x - 12\) See page 204.
17. \(h(x) = \frac{2}{x}\) See page 204.
18. \(k(x) = \frac{1}{x^2}\) See page 204.
19. \(f(x) = e^{2x}\) See page 204.
20. \(f(x) = e^{-0.5x}\) See page 204.
21. \(y = 4 - \sqrt{x + 2}\) See page 204.
22. \(y = x^4 - 10x^2 + 9\) See page 204.

1. (a) Yes, \((b) 2c + 2 = \frac{2}{1 - 0} = \frac{2}{1 - 0} = 3, so c = \frac{1}{2}\)
2. (a) Yes, \((b) \frac{2}{3c^{1/3}} = \frac{1}{1 - 0} = 1, so c = 8\)
3. (a) Yes, \((b) \frac{1}{\sqrt{1 - c^2}} - \frac{(m')^2}{1 - (1)} = \frac{\pi}{2}, so c = \sqrt{1 - \frac{1}{2}} \approx 0.771\)
4. (a) Yes, \((b) \frac{1}{c - 1} = \frac{\ln 3 - \ln 1}{4^2 - 4}, so c = 2.820\).
23. (a) Local max at \((2,6.7, 3.08)\); local min at \((4, 0)\)  
(b) On \((-\infty, 8/3)\)  
(c) On \([8/3, 4)\)  
In Exercises 23–28, find (a) the local extrema, (b) the intervals on which the function is increasing, and (c) the intervals on which the function is decreasing.  

24. (a) Local min at \((-2, -7.56)\)  
(b) \(g(x) = x^{1/3}(x + 8)\)  
(c) On \((-\infty, -2)\)  
25. \(h(x) = \frac{-x}{x^2 + 4}\)  
26. \(k(x) = \frac{x}{x^2 - 4}\)  
27. \(f(x) = x^3 - 2x - 2 \cos x\)  
28. \(g(x) = 2x + \cos x\)  
(a) None  
(b) On \((-\infty, -1)\)  
(c) None  

In Exercises 29–34, find all possible functions \(f\) with the given derivative.  

29. \(f'(x) = x^2 + C\)  
30. \(f'(x) = 2x + C\)  
31. \(f'(x) = 3x^2 - 2x + 1\)  
32. \(f'(x) = \sin x - \cos x + C\)  
33. \(f'(x) = e^x - x^2 + x + C\)  
34. \(f'(x) = \frac{1}{x - 1}, \ x > 1\)  
In Exercises 35–38, find the function with the given derivative whose graph passes through the point \(P\).  

35. \(f'(x) = -\frac{1}{x^2}, \ x > 0\)  
\(P(2, 1)\)  
\(\frac{1}{x} + \frac{1}{2}, x > 0\)  
36. \(f'(x) = \frac{1}{4x^3}, \ P(1, -2)\)  
\(x^{4/3} - 3\)  
37. \(f'(x) = \frac{1}{x + 2}, \ x > -2\)  
\(P(-1, 3)\)  
\(\ln (x + 2) + 3\)  
38. \(f'(x) = 2x + 1 - \cos x, \ P(0, 3)\)  
\(x^2 + x - \sin x + 3\)  

**Group Activity** In Exercises 39–42, sketch a graph of a differentiable function \(y = f(x)\) that has the given properties.  

39. (a) local minimum at \((1, 1)\), local maximum at \((3, 3)\)  
(b) local minima at \((1, 1)\) and \((3, 3)\)  
(c) local maxima at \((1, 1)\) and \((3, 3)\)  
40. \(f(2) = 3, \ f'(2) = 0, \ \text{and}\)  
(a) \(f'(x) > 0 \text{ for } x < 2, \ \ f'(x) < 0 \text{ for } x > 2\)  
(b) \(f'(x) < 0 \text{ for } x < 2, \ \ f'(x) > 0 \text{ for } x > 2\)  
(c) \(f'(x) < 0 \text{ for } x \neq 2\)  
(d) \(f'(x) > 0 \text{ for } x \neq 2\)  
41. \(f'(-1) = f'(1) = 0, \ \ f'(x) > 0 \text{ on } (-1, 1)\)  
\(f'(x) < 0 \text{ for } x < -1, \ \ f'(x) > 0 \text{ for } x > 1\)  
42. A local minimum value that is greater than one of its local maximum values.  
43. **Free Fall** On the moon, the acceleration due to gravity is 1.6 m/sec².  
(a) If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?  
48 m/sec  
(b) How far below the point of release is the bottom of the crevasse?  
720 meters  
(c) If instead of being released from rest, the rock is thrown into the crevasse from the same point with a downward velocity of 4 m/sec, when will it hit the bottom and how fast will it be going when it does?  
After about 27.604 seconds, and it will be going about 48.166 m/sec  
44. **Diving** (a) With what velocity will you hit the water if you step off from a 10-m diving platform?  
14 m/sec  
(b) With what velocity will you hit the water if you dive off the platform with an upward velocity of 2 m/sec?  
10√2 m/sec or, about 14.142 m/sec  
45. **Writing to Learn** The function \(f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}\) is zero at \(x = 0\) and \(x = 1\). Its derivative is equal to 1 at every point between 0 and 1, so \(f'\) is never zero between 0 and 1, and the graph of \(f\) has no tangent parallel to the chord from \((0, 0)\) to \((1, 0)\). Explain why this does not contradict the Mean Value Theorem.  
**Because the function is not continuous on \([0, 1]\).**  
46. **Writing to Learn** Explain why there is a zero of \(y = \cos x\) between every two zeros of \(y = \sin x\).  
47. **Unique Solution** Assume that \(f\) is continuous on \([a, b]\) and differentiable on \((a, b)\). Also assume that \(f(a)\) and \(f(b)\) have opposite signs and \(f' \neq 0\) between \(a\) and \(b\). Show that \(f(x) = 0\) exactly once between \(a\) and \(b\).  
In Exercises 48 and 49, show that the equation has exactly one solution in the interval.  
[Hint: See Exercise 47.]  
48. \(x^4 + 3x + 1 = 0, \ -2 \leq x \leq -1\)  
49. \(x + \ln (x + 1) = 0, \ 0 \leq x \leq 3\)  
50. **Parallel Tangents** Assume that \(f\) and \(g\) are differentiable on \([a, b]\) and that \(f(a) = g(a)\) and \(f(b) = g(b)\). Show that there is at least one point between \(a\) and \(b\) where the tangents to the graphs of \(f\) and \(g\) are parallel or the same line. Illustrate with a sketch.  

**Standardized Test Questions**  
You may use a graphing calculator to solve the following problems.  

51. **True or False** If \(f\) is differentiable and increasing on \((a, b)\), then \(f'(c) > 0\) for every \(c\) in \((a, b)\). Justify your answer.  
52. **True or False** If \(f\) is differentiable and \(f'(c) > 0\) for every \(c\) in \((a, b)\), then \(f\) is increasing on \((a, b)\). Justify your answer.  
51. False. For example, the function \(x^3\) is increasing on \((-1, 1)\), but \(f'(0) = 0\).  
52. True. In fact, \(f\) is increasing on \([a, b]\) by Corollary 1 to the Mean Value Theorem.
53. **Multiple Choice** If \( f(x) = \cos x \), then the Mean Value Theorem guarantees that somewhere between 0 and \( \pi/3 \), \( f'(x) = \) (A) \( \frac{3}{2\pi} \) (B) \( \frac{\sqrt{3}}{2} \) (C) \( -\frac{1}{2} \) (D) 0 (E) \( \frac{1}{2} \)

54. **Multiple Choice** On what interval is the function \( g(x) = e^{5x} \) decreasing? (A) \( (-\infty, 0] \) (B) \([0, 4] \) (C) \([2, 4] \) (D) \((4, \infty) \) (E) no interval

55. **Multiple Choice** Which of the following functions is an antiderivative of \( \frac{1}{\sqrt{x}} \)? (A) \( -\frac{1}{\sqrt{2x}} \) (B) \( \frac{2}{\sqrt{x}} \) (C) \( \frac{\sqrt{x}}{2} \) (D) \( \sqrt{x} + 5 \) (E) \( 2\sqrt{x} - 10 \)

56. **Multiple Choice** All of the following functions satisfy the conditions of the Mean Value Theorem on the interval \([-1, 1]\) \( \text{except:} \) (A) \( \sin x \) (B) \( \sin^{-1} x \) (C) \( 5x^3 \) (D) \( x^3/5 \) (E) \( \frac{x}{x-2} \)

### Explorations

57. **Analyzing Derivative Data** Assume that \( f \) is continuous on \([-2, 2]\) and differentiable on \((-2, 2)\). The table gives some values of \( f'(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f'(x) )</th>
<th>( x )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>7</td>
<td>0.25</td>
<td>-4.81</td>
</tr>
<tr>
<td>-1.75</td>
<td>4.19</td>
<td>0.5</td>
<td>-4.25</td>
</tr>
<tr>
<td>-1.5</td>
<td>1.75</td>
<td>0.75</td>
<td>-3.31</td>
</tr>
<tr>
<td>-1.25</td>
<td>-0.31</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
<td>1.25</td>
<td>-0.31</td>
</tr>
<tr>
<td>-0.75</td>
<td>-3.31</td>
<td>1.5</td>
<td>1.75</td>
</tr>
<tr>
<td>-0.5</td>
<td>-4.25</td>
<td>1.75</td>
<td>4.19</td>
</tr>
<tr>
<td>-0.25</td>
<td>-4.81</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>0</td>
<td>-5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Estimate where \( f \) is increasing, decreasing, and has local extrema.

(b) Find a quadratic regression equation for the data in the table and superimpose its graph on a scatter plot of the data.

(c) Use the model in part (b) for \( f' \) and find a formula for \( f \) that satisfies \( f(0) = 0 \).

58. **Analyzing Motion Data** Priya’s distance \( D \) in meters from a motion detector is given by the data in Table 4.1.

<table>
<thead>
<tr>
<th>( t ) (sec)</th>
<th>( D ) (m)</th>
<th>( t ) (sec)</th>
<th>( D ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>3.36</td>
<td>4.5</td>
<td>3.59</td>
</tr>
<tr>
<td>0.5</td>
<td>2.61</td>
<td>5.0</td>
<td>4.15</td>
</tr>
<tr>
<td>1.0</td>
<td>1.86</td>
<td>5.5</td>
<td>3.99</td>
</tr>
<tr>
<td>1.5</td>
<td>1.27</td>
<td>6.0</td>
<td>3.37</td>
</tr>
<tr>
<td>2.0</td>
<td>0.91</td>
<td>6.5</td>
<td>2.58</td>
</tr>
<tr>
<td>2.5</td>
<td>1.14</td>
<td>7.0</td>
<td>1.93</td>
</tr>
<tr>
<td>3.0</td>
<td>1.69</td>
<td>7.5</td>
<td>1.25</td>
</tr>
<tr>
<td>3.5</td>
<td>2.37</td>
<td>8.0</td>
<td>0.67</td>
</tr>
<tr>
<td>4.0</td>
<td>3.01</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Estimate when Priya is moving toward the motion detector; away from the motion detector.

(b) **Writing to Learn** Give an interpretation of any local extreme values in terms of this problem situation.

(c) Find a cubic regression equation \( D = f(t) \) for the data in Table 4.1 and superimpose its graph on a scatter plot of the data.

(d) Use the model in (c) for \( f \) to find a formula for \( f' \). Use this formula to estimate the answers to (a).

### Extending the Ideas

59. **Geometric Mean** The geometric mean of two positive numbers \( a \) and \( b \) is \( \sqrt{ab} \). Show that for \( f(x) = 1/x \) on any interval \([a, b]\) of positive numbers, the value of \( c \) in the conclusion of the Mean Value Theorem is \( c = \sqrt{ab} \).

60. **Arithmetic Mean** The arithmetic mean of two numbers \( a \) and \( b \) is \((a + b)/2\). Show that for \( f(x) = x^2 \) on any interval \([a, b]\), the value of \( c \) in the conclusion of the Mean Value Theorem is \( c = (a + b)/2 \).

61. **Upper Bounds** Show that for any numbers \( a \) and \( b \), \(| \sin b - \sin a | \leq |b - a | \).

62. **Sign of \( f' \)** Assume that \( f \) is differentiable on \( a \leq x \leq b \) and that \( f(b) < f(a) \). Show that \( f' \) is negative at some point between \( a \) and \( b \).

63. **Monotonic Functions** Show that monotonic increasing and decreasing functions are one-to-one.

Answers:

10. (a) \( y = \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} \), or \( y = 0.707x - 0.707 \)

15. (a) Local maximum at \( \left( \frac{5}{2}, \frac{25}{4} \right) \) (b) On \( \left( -\infty, \frac{5}{2} \right] \)

16. (a) Local minimum at \( \left( \frac{1}{2}, \frac{49}{4} \right) \) (b) On \( \left[ \frac{1}{2}, \infty \right) \)

17. (a) None (b) None (c) On \((-\infty, 0)\) and \((0, \infty)\)

18. (a) None (b) On \((-\infty, 0)\) (c) On \((0, \infty)\)

19. (a) None (b) On \((-\infty, \infty)\) (c) None

20. (a) None (b) None (c) On \((-\infty, \infty)\)

21. (a) Local maximum at \((-2, 4) \) (b) None (c) On \([-2, \infty) \)

22. (a) Local maximum at \((0, 9)\); local minima at \((-\sqrt{5}, -16) \) and \((\sqrt{5}, -16) \) (b) On \([-\sqrt{5}, 0] \) and \([\sqrt{5}, \infty) \)

(e) On \((-\infty, -\sqrt{5})\) and \([0, \sqrt{5})\)